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On the Rate of Convergence of Fourier–Legendre Series of Functions of Bounded Variation

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1. INTRODUCTION

Let $P_n(x)$ be the Legendre polynomial of degree n normalized so that $P_n(1) = 1$. Let f be a function of bounded variation on $[-1, 1]$ and

$$S_n(f, x) = \sum_{k=0}^{n} a_k(f)\, P_k(x)$$

the nth partial sum of the Fourier–Legendre series of f. One has

$$a_k(f) = (k + \tfrac{1}{2}) \int_{-1}^{1} f(t)\, P_k(t)\, dt$$

and

$$S_n(f, x) = \int_{-1}^{1} f(t)\, K_n(x, t)\, dt,$$

where

$$K_n(x, t) = \sum_{k=0}^{n} (k + \tfrac{1}{2})\, P_k(x)\, P_k(t)$$

or

$$K_n(x, t) = \frac{n+1}{2} \left(\frac{P_{n+1}(x)\, P_n(t) - P_{n+1}(t)\, P_n(x)}{x - t} \right).$$

As is well known, the Fourier–Legendre series of a function f of bounded variation on $[-1, 1]$ converges at every point $x \in (-1, 1)$ to $\frac{1}{2}(f(x+0) + (f(x-0))$ (see [1, The Series of Legendre's Coefficients,

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pp. 388–395; 2; 3]). We are interested here in finding an estimate for the rate of convergence of the sequence $S_n(f, x)$ to $\frac{1}{2}(f(x+0)+f(x-0))$. Some results in that direction were obtained in [4, p. 76] for functions of bounded variation which are either continuous or differentiable in a neighborhood of the point x.

The main result of this paper can be stated as follows.

THEOREM 1. *Let f be a function of bounded variation on $[-1, 1]$. Then, for $x \in (-1, 1)$ and $n \geqslant 2$*

$$|S_n(f, x) - \tfrac{1}{2}(f(x+0)+f(x-0))| \leqslant \frac{28(1-x^2)^{-3/2}}{n} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(g_x) + \frac{(1-x^2)^{-1}}{\pi n} |f(x+0) - f(x-0)|, \tag{1.1}$$

where

$$\begin{aligned} g_x(t) &= f(t) - f(x-0), && -1 \leqslant t < x \\ &= 0, && t = x \\ &= f(t) - f(x+0), && x < t \leqslant 1 \end{aligned} \tag{1.2}$$

and $V_a^b(g)$ is the total variation of g on $[a, b]$.

If f is a continuous function of bounded variation the inequality (1.1) becomes

$$|S_n(f, x) - f(x)| \leqslant \frac{28(1-x^2)^{-3/2}}{n} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(f). \tag{1.3}$$

The right-hand side of (1.1) converges to zero as $n \to \infty$ since continuity of $g_x(t)$ at $t = x$ implies that

$$V_{x-\delta}^{x+\delta}(g_x) \to 0 (\delta \to 0+).$$

Results of this type for the Fourier series of a 2π-periodic function of bounded variation on $[-\pi, \pi]$ were proved in [5].

As far as the precision of estimates (1.1) and (1.3) is concerned, we can show that (1.3) cannot be improved asymptotically by considering the Fourier–Legendre expansion of the function $f(x) = |x|^{1/2}$ at $x = 0$. We have for all $x \in (-1, 1)$,

$$f(x) = |x|^{1/2} = 2 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{4m+1}{(4m-1)(4m+3)} P_{2m}(x)$$

and so,

$$S_n(f, 0) - f(0) = 2 \sum_{m=n+1}^{\infty} (-1)^m \frac{4m+1}{(4m-1)(4m+3)} P_{2m}(0).$$

Since

$$P_{2m}(0) = (-1)^m \frac{1.3.5 \ldots . (2m-1)}{2.4.6 \ldots .. (2m)}$$

it follows that

$$\begin{aligned} S_n(f, 0) - f(0) &= 2 \sum_{m=n+1}^{\infty} \frac{4m+1}{(4m-1)(4m+3)} \frac{1.3.5 \ldots \ldots (2m-1)}{2.4.6 \ldots \ldots . (2m)} \\ &\geqslant \sum_{m=n+1}^{\infty} \frac{1}{(4m+3)\sqrt{m}} \\ &\geqslant \frac{1}{7} \sum_{m=n+1}^{\infty} \frac{1}{m^{3/2}} \\ &\geqslant \left(\frac{1}{7\sqrt{2}}\right) \frac{1}{\sqrt{n}}. \end{aligned}$$

On the other hand, from (1.3) follows that

$$| S_n(f, 0) - f(0)| \leqslant \frac{28}{n} \sum_{k=1}^{n} V_{-1/k}^{1/k}(f) \leqslant \frac{56}{n} \sum_{k=1}^{n} V_0^{1/k}(f)$$

Since $V_0^{\delta}(f) = \delta^{1/2}$, we have

$$|S_n(f, 0) - f(0)| \leqslant \frac{56}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant \frac{102}{\sqrt{n}}.$$

Hence, for the function $f(x) = |x|^{1/2}$ we have

$$\frac{1}{7\sqrt{2}\sqrt{n}} \leqslant | S_n(f, 0) - f(0)| \leqslant \frac{102}{\sqrt{n}}.$$

A look at the proof of Theorem 1 shows that the following more general result is true.

THEOREM 2. *Let $K_n(x, t)$ be a continuous function of two variables on $[a, b] \times [a, b]$ and let L_n be the operator which transforms a function f of bounded variation on $[a, b]$ into the function*

$$L_n(f, x) = \int_a^b f(t) K_n(x, t) \, dt, \qquad x \in [a, b].$$

If, for a fixed $x \in (a, b)$ and $n \geqslant 1$, the kernel $K_n(x, t)$ satisfies conditions

(i) $\left| \int_a^x K_n(x, \tau)\, d\tau - \frac{1}{2} \right| \leqslant \frac{A(x)}{n}$ *and* $\left| \int_x^b K_n(x, \tau)\, d\tau - \frac{1}{2} \right| \leqslant \frac{A(x)}{n}$,

(ii) $\int_{x-(x-a)/n}^{x+(b-x)/n} |K_n(x, \tau)|\, d\tau \leqslant B(x)$,

(iii) $\left| \int_a^t K_n(x, \tau)\, d\tau \right| \leqslant \frac{C(x)}{n(x-t)}\ (a \leqslant t < x < b)$ *and*

$\left| \int_t^b K_n(x, \tau)\, d\tau \right| \leqslant \frac{C(x)}{n(t-x)}\ (a < x < t \leqslant b)$,

where $A(x)$, $B(x)$ and $C(x)$ are positive functions on (a, b), then there exists a positive number $M(f, x)$, depending only on f and x, such that

$$|L_n(f, x) - \tfrac{1}{2}(f(x+0) + f(x-0))| \leqslant \frac{M(f, x)}{n} \sum_{k=1}^{n} V_{x-(x-a)/k}^{x+(b-x)/k}(g_x),$$

where, as before,

$$\begin{aligned} g_x(t) &= f(t) - f(x-0), && a \leqslant t < x \\ &= 0, && t = x \\ &= f(t) - f(x+0), && x < t \leqslant b. \end{aligned}$$

2. Lemmas

The proof of Theorem 1 is based on a number of properties of Legendre polynomials. These properties are listed and some of them proved in this section.

Lemma 1. *We have*

$$|P_n(x)| \leqslant \left(\frac{2}{\pi}\right)^{1/2} (1-x^2)^{-1/2} n^{-1/2}, \qquad x \in (-1, 1), \tag{2.1}$$

$$\left| \int_\alpha^\beta P_n(t)\, dt \right| \leqslant \frac{4\sqrt{2\pi}}{(2n+1)(n-1)^{1/2}}, \qquad n \geqslant 2,\ \alpha, \beta \in [-1, 1], \tag{2.2}$$

$$\int_x^1 K_n(x, t)\, dt = \tfrac{1}{2} - \tfrac{1}{2} P_n(x)\, P_{n+1}(x), \tag{2.3}$$

$$\int_{-1}^x K_n(x, t)\, dt = \tfrac{1}{2} + \tfrac{1}{2} P_n(x)\, P_{n+1}(x). \tag{2.4}$$

Proof of Lemma 1. Most of the properties (2.1)–(2.4) are well known. Inequality (2.1) can be found in [4, p. 28] or [6, p. 163]. Inequality (2.2) is a consequence of the inequality

$$\left| \int_x^1 P_n(t)\,dt \right| \leqslant \frac{8}{(2n+1)(2(n-1))^{1/2}} \int_0^\infty e^{-t^2}\,dt,$$

which can be found in [4, p. 72].

As for the proof of (2.3), observe that

$$(2n+1)\,P_n(t) = P'_{n+1}(t) - P'_{n-1}(t)$$

and consequently

$$\begin{aligned}\int_x^1 K_n(x,t)\,dt &= \frac{1}{2} \sum_{k=0}^{n} (2k+1)\,P_k(x) \int_x^1 P_k(t)\,dt \\ &= \frac{1-x}{2} + \frac{1}{2} \sum_{k=1}^{n} P_k(x)(P_{k+1}(t) - P_{k-1}(t))|_x^1.\end{aligned}$$

Since $P_{k+1}(1) - P_{k-1}(1) = 0$, it follows that

$$\begin{aligned}\int_x^1 K_n(x,t)\,dt &= \frac{1-x}{2} - \frac{1}{2} \sum_{k=1}^{n} P_k(x)(P_{k+1}(x) - P_{k-1}(x)) \\ &= \frac{1-x}{2} + \frac{1}{2} \sum_{k=1}^{n} (P_{k-1}(x)\,P_k(x) - P_k(x)\,P_{k+1}(x)) \\ &= \frac{1-x}{2} + \tfrac{1}{2} P_0(x)\,P_1(x) - \tfrac{1}{2} P_n(x)\,P_{n+1}(x).\end{aligned}$$

The proof of formula (2.4) is similar.

LEMMA 2. *For* $x \in (-1, 1)$ *and* $n \geqslant 2$

$$\int_{x-(1+x)/n}^{x+(1-x)/n} |K_n(x,t)|\,dt \leqslant \frac{4}{1-x^2}. \tag{2.5}$$

Proof of Lemma 2. Using (2.1) we find that

$$\begin{aligned}|K_n(x,t)| &= \left| \sum_{k=0}^{n} (k+\tfrac{1}{2})\,P_k(x)\,P_k(t) \right| \\ &\leqslant \frac{1}{2} + \frac{3n}{\pi(1-x^2)^{1/2}(1-t^2)^{1/2}}\end{aligned}$$

and it follows that

$$\int_{x-(1+x)/n}^{x+(1-x)/n} |K_n(x, t)|\,dt$$
$$\leqslant \frac{1}{n} + \frac{3n}{\pi(1-x^2)^{1/2}} \int_{x-(1+x)/n}^{x+(1-x)/n} \frac{dt}{(1-t^2)^{1/2}}. \tag{2.6}$$

To evaluate the integral on the right-hand side of (2.6) suppose first that $0 \leqslant x < 1$. Then

$$\int_{x-(1+x)/n}^{x+(1-x)/n} \frac{dt}{(1-t^2)^{1/2}} = \theta_2 - \theta_1,$$

where $\cos\theta_2 = x - (1+x)/n$, $\cos\theta_1 = x + (1-x)/n$. If $n \geqslant 2$ and $0 \leqslant x < 1$, we have $\cos\theta_2 \geqslant -\frac{1}{2}$, which means that $0 < \theta < 2\pi/3$.

To estimate $\theta_2 - \theta_1$, observe that by the mean-value theorem,

$$\cos\theta_1 - \cos\theta_2 = (\theta_2 - \theta_1)\sin\xi,$$

where $\theta_1 < \xi < \theta_2$.

If $0 < \xi < \pi/3$ and $n \geqslant 2$ we have

$$\begin{aligned} \sin\xi \geqslant \sin\theta_1 &= (1-\cos\theta_1)^{1/2}(1+\cos\theta_1)^{1/2} \\ &\geqslant \left((1-x)\left(1-\frac{1}{n}\right)\right)^{1/2}(1+x)^{1/2} \\ &\geqslant \frac{1}{\sqrt{2}}(1-x^2)^{1/2}. \end{aligned}$$

If $\pi/3 \leqslant \xi \leqslant 2\pi/3$, we have

$$\sin\xi \geqslant \frac{\sqrt{3}}{2} \geqslant \frac{1}{\sqrt{2}} \geqslant \frac{1}{\sqrt{2}}(1-x^2)^{1/2}.$$

Consequently,

$$\frac{2}{n} = \cos\theta_1 - \cos\theta_2 \geqslant \frac{1}{\sqrt{2}}(1-x^2)^{1/2}(\theta_2 - \theta_1)$$

or

$$\theta_2 - \theta_1 \leqslant \frac{2\sqrt{2}}{n}(1-x^2)^{-1/2}.$$

Hence

$$\int_{x-(1+x)/n}^{x+(1-x)/n} \frac{dt}{(1-t^2)^{1/2}} \leqslant \frac{2\sqrt{2}}{n}(1-x^2)^{-1/2},$$

and (2.5) follows from this inequality and (2.6) if $0 \leqslant x \leqslant 1$ and $n \geqslant 2$.
If $-1 < x \leqslant 0$,

$$\int_{x-(1+x)/n}^{x+(1-x)/n} |K_n(x,t)|\,dt = \int_{-|x|-(1-|x|)/n}^{-|x|+(1+|x|)/n} |K_n(-|x|,t)\,dt.$$

Since $K_n(-x,t) = K_n(x,-t)$ we have

$$\int_{x-(1+x)/n}^{x+(1-x)/n} |K_n(x,t)|\,dt = \int_{-|x|-(1-|x|)/n}^{-|x|+(1+|x|)/n} |K_n(|x|,-t)|\,dt$$

$$= \int_{|x|-(1+|x|)/n}^{|x|+(1-|x|)/n} |K_n(|x|,t)|\,dt$$

and (2.5) follows again

LEMMA 3. *For $-1 \leqslant t < x < 1$ and $n \geqslant 2$.*

$$\left| \int_{-1}^{t} K_n(x,\tau)\,d\tau \right| \leqslant \frac{6}{n(x-t)}(1-x^2)^{-1/2} \tag{2.7}$$

and for $-1 < x < t \leqslant 1$ and $n \geqslant 2$

$$\left| \int_{t}^{1} K_n(x,\tau)\,d\tau \right| \leqslant \frac{6}{n(t-x)}(1-x^2)^{-1/2}. \tag{2.8}$$

Proof of Lemma 3. Since

$$K_n(x,\tau) = \frac{n+1}{2}\left(\frac{P_{n+1}(x)\,P_n(\tau) - P_n(x)\,P_{n+1}(\tau)}{x-\tau}\right)$$

and $1/(x-\tau)$ for fixed $x \in (-1,1)$ is an increasing function of τ on $[-1,t]$, $-1 < t < x$, we find, by the mean-value theorem, that

$$\int_{-1}^{t} K_n(x,\tau)\,d\tau = \frac{n+1}{2}\,\frac{1}{x-t}\left(P_{n+1}(x)\int_{\xi}^{t} P_n(\tau)\,d\tau - P_n(x)\int_{\xi}^{t} P_{n+1}(\tau)\,d\tau\right).$$

Now, using inequalities (2.1) and (2.2), we get

$$\left|\int_{-1}^{t} K_n(x,\tau)\,d\tau\right| \leqslant \frac{n+1}{2}\cdot\frac{1}{x-t}\left(\left(\frac{2/\pi}{n+1}\right)^{1/2}\cdot\frac{4(2\pi)^{1/2}}{2n+1}(n-1)^{-1/2}\right.$$
$$\left.+\left(\frac{2/\pi}{n}\right)^{1/2}\cdot\frac{4(2\pi)^{1/2}}{2n+3}n^{-1/2}\right)(1-x^2)^{-1/2}$$
$$\leqslant \frac{n+1}{2}\cdot\frac{8}{x-t}\left(\frac{(n+1)^{-1/2}(n-1)^{-1/2}}{(2n+1)}\right.$$
$$\left.+\frac{1}{(2n+3)n}\right)(1-x^2)^{-1/2}.$$

Since $n-1\geqslant(n+1)/3$ for $n\geqslant 2$, it follows that

$$\left|\int_{-1}^{t} K_n(x,\tau)\,d\tau\right| \leqslant \frac{2(1+\sqrt{3})}{(x-t)n}(1-x^2)^{-1/2}$$

and (2.7) follows.

The proof of (2.8) is similar.

3. Proof of Theorem 1

For any fixed $x\in(-1,1)$ we have

$$S(f,x)=\int_{-1}^{1} f(t)\,K_n(x,t)\,dt$$
$$=\int_{-1}^{x}(f(t)-f(x-0))\,K_n(x,t)\,dt+\int_{x}^{1}(f(t)-f(x+0))\,K_n(x,t)\,dt$$
$$+f(x-0)\int_{-1}^{x}K_n(x,t)\,dt+f(x+0)\int_{x}^{1}K_n(x,t)\,dt.$$

Using (1.2), (2.3) and (2.4), this equality becomes

$$S_n(f,x)=\tfrac{1}{2}(f(x-0)+f(x+0))$$
$$+\int_{-1}^{1} g_x(t)\,K_n(x,t)\,dt-\tfrac{1}{2}(f(x+0)-f(x-0))\,P_n(x)\,P_{n+1}(x).$$

Hence

$$|S_n(f, x) - \tfrac{1}{2}(f(x+0) + f(x-0))|$$

$$\leqslant \left| \int_{-1}^{1} g_x(t)\, K_n(x, t)\, dt \right| + \tfrac{1}{2}|f(x+0) - f(x-0)|\, |P_n(x)\, P_{n+1}(x)|. \tag{3.1}$$

For the second term on the right-hand side of inequality (3.1) we have by (2.1)

$$\tfrac{1}{2}|f(x+0) - f(x-0)|\, |P_n(x)\, P_{n+1}(x)| \leqslant \frac{1}{n\pi} |f(x+0) - f(x-0)|(1 - x^2)^{-1}.$$

Hence, Theorem 1 will be proved if we establish that

$$\left| \int_{-1}^{1} g_x(t)\, K_n(x, t) \right| \leqslant \frac{28(1 - x^2)^{-3/2}}{n} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(g_x) \tag{3.2}$$

for all $n \geqslant 2$ and $x \in (-1, 1)$.

To do this we first decompose the integral on the left-hand side of (3.2) in three parts, as follows.

$$\int_{-1}^{1} g_x(t)\, K_n(x, t)\, dt$$

$$= \left(\int_{-1}^{x-(1+x)/n} + \int_{x-(1+x)/n}^{x+(1-x)/n} + \int_{x+(1-x)/n}^{1} \right) g_x(t)\, K_n(x, t)\, dt$$

$$= A_n(f, x) + B_n(f, x) + C_n(f, x). \tag{3.3}$$

The evaluation of the middle term is easy in view of Lemma 2. For $t \in [x - (1+x)/n, x + (1-x)/n]$,

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V_{x-(1+x)/n}^{x+(1-x)/n}(g_x),$$

and so

$$|B_n(f, x)| = \left| \int_{x-(1+x)/n}^{x+(1-x)/n} g_x(t)\, K_n(x, t)\, dt \right|$$

$$\leqslant V_{x-(1+x)/n}^{x+(1-x)/n}(g_x) \int_{x-(1+x)/n}^{x+(1-x)/n} |K_n(x, t)|\, dt.$$

Using Lemma 2, we find that

$$|B_n(f, x)| \leqslant \frac{4}{1 - x^2} V_{x-(1+x)/n}^{x+(1-x)/n}(g_x). \tag{3.4}$$

The evaluations of $A_n(f, x)$ and $C_n(f, x)$ are similar. In the first case let us denote

$$y = x - \frac{1+x}{n} \text{ and } \lambda_n(x, t) = \int_{-1}^{t} K_n(x, \tau)\, d\tau.$$

We have then

$$A_n(f, x) = \int_{-1}^{y} g_x(t)\, K_n(x, t)\, dt = \int_{-1}^{y} g_x(t)\, d\lambda_n(x, t).$$

By partial integration

$$A_n(f, x) = g_x(y)\, \lambda_n(x, y) - \int_{-1}^{y} \lambda_n(x, t)\, dg_x(t).$$

Hence

$$|A_n(f, x)| \leqslant |g_x(y)|\, |\lambda_n(x, y)| + \int_{-1}^{y} |\lambda_n(x, t)|\, d(-V_t^x(g_x)).$$

Using the fact that

$$|g_x(y)| = |g_x(y) - g_x(x)| \leqslant V_y^x(g_x)$$

and that by Lemma 3,

$$|\lambda_n(x, t)| \leqslant \frac{6}{n(x-t)} (1-x^2)^{1/2} \qquad \text{for} \quad -1 \leqslant t \leqslant y < x,$$

we find that

$$|A_n(f, x)| \leqslant \frac{6(1-x^2)^{-1/2}}{n} \left(\frac{1}{x-y} V_y^x(g_x) + \int_{-1}^{y} \frac{1}{x-t} d(-V_t^x(g_x)) \right).$$

Since

$$\int_{-1}^{y} \frac{1}{x-t} d(-V_t^x(g_x)) = -\frac{1}{x-t} V_t^x(g_x)\Big|_{-1}^{y} + \int_{-1}^{y} V_t^x(g_x) \frac{dt}{(x-t)^2},$$

it follows that

$$|A_n(f, x)| \leqslant \frac{6(1-x^2)^{-1/2}}{n} \left(\frac{1}{1+x} V_{-1}^x(g_x) + \int_{-1}^{x-(1+x)/n} V_t^x(g_x) \frac{dt}{(x-t)^2} \right).$$

Replacing the variable t in the last integral by $x-(1+x)/t$ we find that

$$\int_{-1}^{x-(1+x)/n} V_t^x(g_x)\frac{dt}{(x-t)^2}=\frac{1}{1+x}\int_1^n V_{x-(1+x)/t}^x(g_x)\,dt$$
$$\leqslant\frac{1}{1+x}\sum_{k=1}^{n-1} V_{x-(1+x)/k}^x(g_x)$$

and so

$$|A_n(f,x)|\leqslant\frac{12}{n(1+x)}(1-x^2)^{-1/2}\sum_{k=1}^{n-1} V_{x-(1+x)/k}^x(g_x)$$
$$\leqslant\frac{24}{n}(1-x^2)^{-3/2}\sum_{k=1}^{n} V_{x-(1+x)/k}^x(g_x). \tag{3.5}$$

In order to evaluate $C_n(f,x)$, let $z=x+(1-x)/n$ and $\Lambda_n(x,t)=\int_t^1 K_n(x,\tau)\,d\tau$. We have then

$$C_n(f,x)=\int_z^1 g_x(t)\,K_n(x,t)\,dt=-\int_z^1 g_x(t)\,d\Lambda_n(x,t).$$

Using partial integration we find that

$$C_n(f,x)=g_x(z)\,\Lambda_n(x,z)+\int_z^1 \Lambda_n(x,t)\,dg_x(t)$$

so that

$$|C_n(f,x)|\leqslant|g_x(z)|\,|\Lambda_n(x,z)|+\int_z^1|\Lambda_n(x,t)|\,dV_x^t(g_x).$$

Since

$$|g_x(z)|=|g_x(z)-g_x(x)|\leqslant V_x^z(g_x),$$

and, by Lemma 3,

$$|\Lambda_n(x,t)|\leqslant\frac{6}{n(t-x)}(1-x^2)^{-1/2}\qquad\text{for}\quad x<t\leqslant 1,$$

we find that

$$|C_n(f,x)|\leqslant\frac{6}{n}(1-x^2)^{-1/2}\left(\frac{1}{z-x}V_x^z(g_x)+\int_z^1\frac{1}{t-x}\,dV_x^t(g_x)\right).$$

Using partial integration again, we see that

$$\int_z^1 \frac{1}{t-x}\, dV_x^1(g_x) = \frac{1}{t-x} V_x^t(g_x)\Big|_z^1 + \int_z^1 V_x^t(g_x) \frac{dt}{(t-x)^2},$$

and the preceding inequality becomes

$$|C_n(f, x)| \leqslant \frac{6}{n}(1-x^2)^{-1/2} \left(\frac{1}{1-x} V_x^1(g_x) + \int_{x+(1-x)/n}^1 V_x^t(g_x) \frac{dt}{(t-x)^2} \right).$$

Replacing the variable t in the last integral by $x + (1-x)/t$, we find that

$$\int_{x+(1-x)/n}^1 V_x^t(g_x) \frac{dt}{(t-x)^2} = \frac{1}{1-x} \int_1^n V_x^{x+(1-x)/t}(g_x)\, dt$$
$$\leqslant \frac{1}{1-x} \sum_{k=1}^{n-1} V_x^{x+(1-x)/k}(g_x).$$

Using this inequality we get

$$|C_n(f, x)| \leqslant \frac{12}{n(1-x)}(1-x^2)^{-1/2} \sum_{k=1}^{n-1} V_x^{x+(1-x)/k}(g_x)$$
$$\leqslant \frac{24}{n}(1-x^2)^{-3/2} \sum_{k=1}^{n} V_x^{x+(1-x)/k}(g_x). \tag{3.6}$$

Finally, from (3.3), (3.4), (3.5) and (3.6), we obtain

$$\left| \int_{-1}^1 g_x(t)\, K_n(x, t)\, dt \right| \leqslant \frac{4}{1-x^2} V_{x-(1+x)/n}^{x+(1-x)/n}(g_x)$$
$$+ \frac{24}{n}(1-x^2)^{-3/2} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(g_x).$$

Inequality (3.2) then follows, since $(1-x^2)^{-1} \leqslant (1-x^2)^{-3/2}$ and

$$V_{x-(1+x)/n}^{x+(1-x)/n}(g_x) \leqslant \frac{1}{n} \sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(g_x).$$

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